# Geometric Rotation

There are four methods to represent a rotation: Euler angle, Quaternion Rotation vector/matrix.

## Euler angle

The rotations around Cartesian coordinate axis *X, Y,* *Z* are usually called Yaw, Pitch and Roll, which are note as Euler angle of *α*, *β*, *γ* in equations bellow. There are 12 types of Euler angle rotational sequence, including 6 types of Tait-Bryan Angle (*XYZ, XZY, YXZ, YZX, ZXY, ZYX*), and 6 types of Proper Euler Angle (*XYX, YXY, XZX, ZXZ, YZY, ZYZ*). In the following, we will mainly discuss the rotations of Euler angle sequenced *XYZ*.

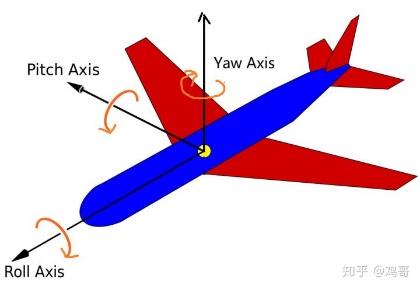


Figure 1‑1 Z-Roll，Y-Pitch，X-Yaw

## Quaternion

Irish mathematician Sir William Rowan Hamilton (1805-1865) discovered the algebra of quaternions, and found that imaginary elements of *i, j, k* can be used to express the unit vectors *x, y, z* of Cartesian coordinate systems. Quaternion is defined as below.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
&\boldsymbol{q}=w+xi+yj+zk, \textrm{note as:}[w,x,y,z]^T \\
&||\boldsymbol{q}||=w^2+x^2+y^2+z^2=1
\end{aligned}
\]
\end{document}

(1‑1)

Quaternion represent rotation around axis %FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
[cos\phi_x, cos\phi_y, cos\phi_z]
\]
\end{document} with angle *θ*.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\left\{\begin{aligned}
&w=cos(\theta/2)\\
&x=sin(\theta/2)cos(\phi_x)\\ 
&y=sin(\theta/2)cos(\phi_y)\\
&z=sin(\theta/2)cos(\phi_z)
\end{aligned}
\right.
\]
\end{document}

(1‑2)

The imaginary unit *i, j, k* conforms to the following equation.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
&i^2=j^2=k^2=ijk=-1\\
&i^{-1}=-i,j^{-1}==-j,k^{-1}=-k\\
&ij=k \rightarrow ji=-k\\
&jk=i \rightarrow kj=-i\\
&ki=j \rightarrow ik=-j
\end{aligned}
\]
\end{document}

(1‑3)

Which is similar with the operation of cross product of Cartesian coordinate unit vector ***x****,* ***y****,* ***z***.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
&\boldsymbol{x}\times\boldsymbol{y}=\boldsymbol{z}
\rightarrow 
\boldsymbol{y}\times\boldsymbol{x}=-\boldsymbol{z}\\
&\boldsymbol{y}\times\boldsymbol{z}=\boldsymbol{x}
\rightarrow
\boldsymbol{z}\times\boldsymbol{y}=-\boldsymbol{x}\\
&\boldsymbol{z}\times\boldsymbol{x}=\boldsymbol{y}
\rightarrow
\boldsymbol{x}\times\boldsymbol{z}=-\boldsymbol{y}
\end{aligned}
\]
\end{document} %FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[,\textrm{Unit vector:}
\left\{
\begin{matrix}
\boldsymbol{x}=[1,0,0]^T\\ 
\boldsymbol{y}=[0,1,0]^T\\ 
\boldsymbol{z}=[0,0,1]^T
\end{matrix}
\right.
\]
\end{document}

(1‑4)

The add and product operation between quaternions as below.

 %FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
\boldsymbol{q}_1+\boldsymbol{q}_2 &=
[w_1+w_2,x_1+x_2,y_1+y_2,z_1+z_2]^T\\
\boldsymbol{q}_1\boldsymbol{q}_2 &=
(w_1 + x_1 i + y_1 j + z_1 k)
(w_2 + x_2 i + y_2 j + z_2 k)\\ 
&=\begin{bmatrix}
w_1w_2-x_1x_2-y_1y_2-z1_z2\\ 
w_1x_2+x_1w_2+y_1z_2-z1_y2\\
w_1y_2-x_1z_2+y_1w_2+z1_x2\\ 
w_1z_2+x_1y_2-y_1x_2+z1_w2\\ 
\end{bmatrix}
\end{aligned}
\]
\end{document}

(1‑5)

## Rotation vector

Rotation vector ***r*** *=* [*x, y, z*] is used to indicate direction of rotation axis ***k***, and vector length equal to the rotation angle *θ* = ||***r***||.

Figure 1‑2 Rotation vector definition

## Rotation matrix

Rotation matrix is the most widely used method to implement a rotation by using a matrix multiplication, such as rotations of Euler angle as below.

%FontSize=8
%TeXFontSize=8
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{R}_\alpha=
\begin{bmatrix}
 1&  0&  0& 0\\ 
 0&  cos_\alpha&  -sin_\alpha& 0\\ 
 0&  sin_\alpha& cos_\alpha& 0\\ 
 0&  0&  0& 1
\end{bmatrix},
\boldsymbol{R}_\beta=
\begin{bmatrix}
 cos_\beta&  0&  sin_\beta& 0\\ 
 0&  1&  0&  0\\ 
 -sin_\beta& 0&  cos_\beta& 0\\ 
 0&  0&  0& 1
\end{bmatrix},
\boldsymbol{R}_\gamma=
\begin{bmatrix} 
 cos_\gamma&  -sin_\gamma  &0  &0\\ 
 sin_\gamma& cos_\gamma  &0  &0\\ 
 0&  0&  1& 0\\ 
 0&  0&  0& 1
\end{bmatrix}
\]
\end{document}

(1‑6)

There are several transformations between Euler angle, quaternion and rotation vector / matrix, as shown in the following figure.



Figure 1‑3 Transformations of rotation presentations

## (1-4). Euler angle(x-y-z order) to rotation matrix

%FontSize=9
%TeXFontSize=9
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
 \boldsymbol{R}_{\alpha\beta\gamma}&=
\boldsymbol{R}_\gamma
\boldsymbol{R}_\beta
\boldsymbol{R}_\alpha
\\ 
 &=\begin{bmatrix}
 cos_\gamma cos_\beta&
 cos_\gamma sin_\beta sin_\alpha -sin_\gamma cos_\alpha &
 cos_\gamma sin_\beta cos_\alpha +sin_\gamma sin_\alpha &  0\\ 
 sin_\gamma cos_\beta&
 sin_\gamma sin_\beta sin_\alpha +cos_\gamma cos_\alpha &
 sin_\gamma sin_\beta cos\alpha - cos_\gamma sin_\alpha & 0\\ 
 -sin_\beta & cos_\beta sin_\alpha & cos_\beta cos_\alpha & 0\\ 
 0&  0&  0& 1
\end{bmatrix}
\end{aligned}
\]
\end{document}

(1‑7)

## (4-1). Rotation matrix to Euler angle(x-y-z order)

Notice that every element of the matrix is a function of pitch angle *β*. There will be a problem named Gimbal Lock if *β* is closed to ±π/2. Formula (1‑6) could be simplified if *β* ≈ π/2.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
 \boldsymbol{R}_{\alpha\beta\gamma}&=
\boldsymbol{R}_\gamma
\boldsymbol{R}_\beta
\boldsymbol{R}_\alpha
\\ 
 &=\begin{bmatrix}
 0& -sin_(\gamma -\alpha)& cos_(\gamma -\alpha)& 0\\ 
 0& cos_(\gamma -\alpha)& sin_(\gamma -\alpha)& 0\\ 
 -1& 0 & 0 & 0\\ 
 0&  0&  0& 1
\end{bmatrix}
\end{aligned}
\]
\end{document}

(1‑8)

Else if *β* ≈ -π/2.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
 \boldsymbol{R}_{\alpha\beta\gamma}&=
\boldsymbol{R}_\gamma
\boldsymbol{R}_\beta
\boldsymbol{R}_\alpha
\\ 
 &=\begin{bmatrix}
 0& -sin_(\gamma +\alpha)& -cos_(\gamma +\alpha)& 0\\ 
 0& cos_(\gamma +\alpha)& -sin_(\gamma +\alpha)& 0\\ 
 1& 0 & 0 & 0\\ 
 0&  0&  0& 1
\end{bmatrix}
\end{aligned}
\]
\end{document}

(1‑9)

Therefore Euler angles can be solved as below.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{cases}
 &\beta=atan2(sin_\beta,cos_\beta),\textrm{where}:sin_\beta=-r_{31},cos_\beta=\sqrt{r_{32}^2+r_{33}^2}\\ 
 &\textrm{if }cos_\beta>10^{-6}:\qquad \alpha=atan2(r_{32},r_{33}),
\gamma=atan2(r_{21},r_{11})\\
 &\textrm{else if }sin_\beta>0:\quad (\alpha-\gamma)=atan2(r_{12},r_{13})\\
 &\textrm{else}:\qquad\qquad\qquad(\alpha+\gamma)=atan2(-r_{12},-r_{13})\\
\end{cases}
\]
\end{document}

(1‑10)

## (2-4). Quaternion to rotation matrix

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{R}=
\begin{bmatrix}
 w^2+x^2-y^2-z^2&  2xy-2zw&  2xz+2yw\\ 
 2xy+2zw&  w^2-x^2+y^2-z^2&  2yz-2xw\\  
 2xz-2yw&  2yz+2xw& w^2-x^2-y^2+z^2
\end{bmatrix}
\]
\end{document}

(1‑11)

## (4-2). Rotation matrix to quaternion

Therefor the rotations around Cartesian coordinate axis *Z, X, Y* could be represent as quaternions of ***q****x,* ***q****y* and ***q****z*.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\left\{
\begin{aligned}
 w&= \frac{\sqrt{r_{11}+r_{22}+r_{32}+1}}{2}\vspace{1ex} \\
 x&= \frac{r_{32}-r_{23}}{4w}\vspace{1ex}\\
 y&= \frac{r_{13}-r_{31}}{4w}\vspace{1ex}\\
 z&= \frac{r_{21}-r_{12}}{4w}\vspace{1ex}
\end{aligned}
\right.
\]
\end{document}

(1‑12)

The result will be unstable if *w* problem still around Cartesian coordinate axis *Z, X, Y* could be represent as quaternions of ***q****x,* ***q****y* and ***q****z*.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\left\{
\begin{aligned}
 w&= \frac{\sqrt{r_{11}+r_{22}+r_{33}+1}}{2}\vspace{1ex} \\
 x&= \sqrt{\frac{r_{11}+1-2w^2}{2}}\vspace{1ex} \\
 y&= \sqrt{\frac{r_{22}+1-2w^2}{2}}\vspace{1ex}\\
 z&= \sqrt{\frac{r_{33}+1-2w^2}{2}}\vspace{1ex}
\end{aligned}
\right.
\]
\end{document}

## (3-4). Rotation vector to rotation matrix

Rodrigues transform: in rotation from vector ***v*** to ***v***rot , the vector ***r*** *=* [*x, y, z*] represent the rotation around the axis of vector ***k***, with rotation angles of ||***r***||.

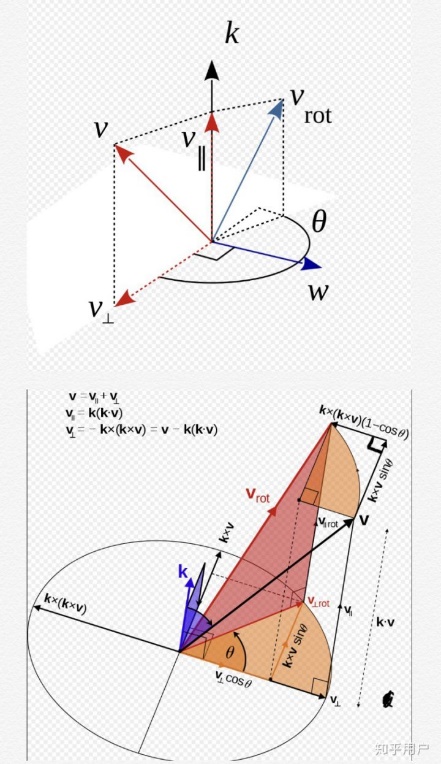


Figure 1‑5 Rodrigues transform from Rotation vector to matrix

Get rotation axis ***k*** and angles *θ* from rotation vector ***r***.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\theta=\| \boldsymbol{r} \|,\quad
\boldsymbol{k}=\boldsymbol{r}/\theta
\]
\end{document}

(1‑13)

Decompose vector ***v*** into vertical & parallel components with axis ***k***.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
&\boldsymbol{v}=\boldsymbol{v}_\perp+\boldsymbol{v}_\parallel,\\
&\boldsymbol{v}_\parallel=(\boldsymbol{v}\cdot\boldsymbol{k})k, \\
&\boldsymbol{v}_\perp=\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{k}=
-\boldsymbol{k}\times (\boldsymbol{k}\times\boldsymbol{v})
\end{aligned}
\]
\end{document}

(1‑14)

Right handed chiral vector ***w*** with perpendicular to %FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{v}_\perp \textrm{ and } \boldsymbol{v}_\parallel
\]
\end{document}

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{w}=\boldsymbol{k}\times \boldsymbol{v}_\perp=
\boldsymbol{k}\times (\boldsymbol{v}-\boldsymbol{v}_\parallel)=
\boldsymbol{k}\times \boldsymbol{v}
\]
\end{document}

(1‑15)

***K*** matrix is used to replace operation ***k***×***v*** by inner of ***Kv***.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{k}\times \boldsymbol{v}=
\begin{bmatrix}
 k_y v_z - k_z v_y\\ 
 k_z v_x - k_x v_z\\
 k_x v_y - k_y v_x
\end{bmatrix}=
\begin{bmatrix}
 0&  -k_z&  k_y\\ 
 k_z&  0&  -k_x\\  
 -k_y&  k_x& 0
\end{bmatrix}
\begin{bmatrix}
 v_x\\ 
 v_y\\ 
 v_z
\end{bmatrix}
=\boldsymbol{K} \boldsymbol{v}
\]
\end{document}

(1‑16)

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{k}\times (\boldsymbol{k}\times \boldsymbol{v})=
\boldsymbol{K}^2 \boldsymbol{v}=
\boldsymbol{K}(\boldsymbol{K} \boldsymbol{v})
\]
\end{document}

(1‑17)

In rotation from vector ***v*** to ***v***rot:

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{v}_{rot\parallel}=\boldsymbol{v}_\parallel, \quad
|\boldsymbol{v}_{rot\perp}|=|\boldsymbol{v}_\perp|, \quad
\boldsymbol{v}_{rot\perp}=cos\theta\boldsymbol{v}_\perp + sin\theta \boldsymbol{w}
\]
\end{document}

(1‑18)

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
\boldsymbol{v}_{rot}
 &= \boldsymbol{v}_{rot\parallel}+
     \boldsymbol{v}_{rot\perp}
 = \boldsymbol{v}_\parallel+
     cos\theta \boldsymbol{v}_\perp+
     sin\theta \boldsymbol{k}\times\boldsymbol{v}\\
 &=(\boldsymbol{k}\cdot     
    \boldsymbol{v})\boldsymbol{k}+
     cos\theta [\boldsymbol{v}-
     (\boldsymbol{k}\cdot   
    \boldsymbol{v})\boldsymbol{k}]+
     sin\theta \boldsymbol{k}\times\boldsymbol{v}\\
 &=\boldsymbol{v}+
    (cos\theta-1)\boldsymbol{v}-
    (cos\theta-1) (\boldsymbol{k}\cdot 
    \boldsymbol{v})\boldsymbol{k}+
    sin\theta \boldsymbol{k}\times\boldsymbol{v}\\
 &=\boldsymbol{v}+
    (1-cos\theta)\boldsymbol{k}\times  
    (\boldsymbol{k}\times \boldsymbol{v})+
    sin\theta \boldsymbol{k}\times\boldsymbol{v}\\
 &=\boldsymbol{v}+
    (1-cos\theta)\boldsymbol{K}^2\boldsymbol{v}+
    sin\theta \boldsymbol{K}\boldsymbol{v}
\end{aligned}
\]
\end{document}

(1‑19)

Rotation matrix ***R***, since ***v***rot = ***Rv***:

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
\boldsymbol{R}
  &=\boldsymbol{I}+
    (1-cos\theta)\boldsymbol{K}^2+
    sin\theta \boldsymbol{K}
\end{aligned}
\]
\end{document}

(1‑20)

In which ***K***:

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{K}=
\begin{bmatrix}
 0&  -k_z&  k_y\\ 
 k_z&  0&  -k_x\\  
 -k_y&  k_x& 0
\end{bmatrix}
\]
\end{document}

(1‑21)

## (4-3). Rotation matrix to rotation vector

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{K}'=
\frac{\boldsymbol{R}-\boldsymbol{R}^T}{2}=
sin\theta
\begin{bmatrix}
 0&  -k_z&  k_y\\ 
 k_z&  0&  -k_x\\  
 -k_y&  k_x& 0
\end{bmatrix}
\]
\end{document}

(1‑22)

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\boldsymbol{r}=sgn(k'_{3,2})
\frac{sin^{-1}\left ( \sqrt{\sum{k'}_{ij}^2/2})\right )} {\sqrt{\sum{k'}_{ij}^2/2}}
[{k'}_{3,2}, {k'}_{1,3}, {k'}_{2,1}]
\]
\end{document}

(1‑23)

## (1-2). Euler angle(x-y-z order) to quaternion

The rotations around Cartesian coordinate axis *X, Y, Z* could be represent as quaternions of ***q****x,* ***q****y* and ***q****z*.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{matrix}
\boldsymbol{q}_x=cos\frac{\alpha}{2}+sin\frac{\alpha}{2}i=
[cos\frac{\alpha}{2},sin\frac{\alpha}{2},0,0]^T \vspace{1ex}\\
\boldsymbol{q}_y=cos\frac{\beta}{2}+sin\frac{\beta}{2}j=
[cos\frac{\beta}{2},0,sin\frac{\beta}{2},0]^T \vspace{1ex}\\
\boldsymbol{q}_x=cos\frac{\gamma}{2}+sin\frac{\gamma}{2}k=
[cos\frac{\gamma}{2},0,0,sin\frac{\gamma}{2}]^T
\end{matrix}
\]
\end{document}

(1‑24)

Therefor quaternion of x-y-z ordered Euler angle rotation note as ***q****xyz* could be solved as below.

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
\boldsymbol{q}_{xyz} &=\begin{bmatrix}cos\frac{\alpha}{2}\vspace{1ex}\\ sin\frac{\alpha}{2}\vspace{1ex}\\ 0\vspace{1ex}\\ 0\end{bmatrix}
\begin{bmatrix}cos\frac{\beta}{2}\vspace{1ex}\\ 0\vspace{1ex}\\ sin\frac{\beta}{2}\vspace{1ex}\\ 0\end{bmatrix}
\begin{bmatrix}cos\frac{\gamma}{2}\vspace{1ex}\\ 0\vspace{1ex}\\ 0\vspace{1ex}\\ sin\frac{\gamma}{2}\end{bmatrix}\\
&=\begin{bmatrix}
cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2}-sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}\vspace{1ex}\\
sin\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2}+cos\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}\vspace{1ex}\\ 
cos\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\gamma}{2}-sin\frac{\alpha}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2}\vspace{1ex}\\
sin\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\gamma}{2}+cos\frac{\alpha}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2}
\end{bmatrix}
\end{aligned}
\]
\end{document}

(1‑25)

## (2-1). Quaternion to Euler angle(x-y-z order)

According to formula (1‑25), the relationship between Euler angle and quaternion is as follows.

%FontSize=9
%TeXFontSize=9
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{matrix}
2w^2+2x^2=1+cos\beta cos\gamma &2w^2-2y^2=cos\beta(cos\alpha+cos\gamma)\ &2w^2+2z^2=1+cos\alpha cos\beta\\
2x^2+2y^2=1 - cos\alpha cos\beta &2y^2+2z^2=1 - cos\beta cos\gamma  &2x^2-2z^2=cos\beta(cos\gamma-cos\alpha)\\
2wz-2xy=cos\beta sin\gamma &2wx-2yz=sin\alpha cos\beta &2wy+2xz=sin\beta\\
\end{matrix}
\]
\end{document}

(1‑26)

Rotations around Cartesian coordinate axis *X, Y, Z* could be represent as

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{bmatrix}
\alpha\\ \beta \\ \gamma
\end{bmatrix}=
\begin{bmatrix}
atan2(2wx-2yz,1-2x^2-2y^2)\\
asin(2wy+2xz)\\
atan2(2wz-2xy,1-2y^2-2z^2)
\end{bmatrix}
\]
\end{document}

(1‑27)

Gimbal Lock problem still exist if *cosβ*=0.

## (2-3). Quaternion to rotation vector

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
\theta &=2\cdot acos(w)\\
\boldsymbol{r}&=\theta \frac{[x,y,z]}{\sqrt{1-w^2}}
\end{aligned}
\]
\end{document}

(1‑28)

## (3-2). Rotation vector to quaternion

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
w&=cos(||\boldsymbol{r}||/2)\\
\begin{bmatrix}
x\\ y\\ z
\end{bmatrix} &= sin(||\boldsymbol{r}||/2)\boldsymbol{r}
\end{aligned}
\]
\end{document}

(1‑29)

%FontSize=10
%TeXFontSize=10
\documentclass{article}
\pagestyle{empty}
\begin{document}
\[
\begin{aligned}
\boldsymbol{q}_{xyz} &=\begin{bmatrix}cos\frac{\alpha}{2}\vspace{1ex}\\ sin\frac{\alpha}{2}\vspace{1ex}\\ 0\vspace{1ex}\\ 0\end{bmatrix}
\begin{bmatrix}cos\frac{\beta}{2}\vspace{1ex}\\ 0\vspace{1ex}\\ sin\frac{\beta}{2}\vspace{1ex}\\ 0\end{bmatrix}
\begin{bmatrix}cos\frac{\gamma}{2}\vspace{1ex}\\ 0\vspace{1ex}\\ 0\vspace{1ex}\\ sin\frac{\gamma}{2}\end{bmatrix}\\
&=\begin{bmatrix}
cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2}-sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}\vspace{1ex}\\
sin\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\gamma}{2}+cos\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}\vspace{1ex}\\ 
cos\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\gamma}{2}-sin\frac{\alpha}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2}\vspace{1ex}\\
sin\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\gamma}{2}+cos\frac{\alpha}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2}
\end{bmatrix}
\end{aligned}
\]
\end{document}

## (…). The rest conversions

Based on the above knowledge, it is easy to solve Euler angle from quaternion by intermediately transform with rotation matrix, the rests follow similar path.